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Contraction Rule of q-Deformed Levi-Civita Symbol

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In this paper we obtain the q-analog of the contraction rule of the q-deformed Levi-Civita symbol and prove it. We use this to present the simplest example of q-vector formula for the q-outer product.

The quantum Yang–Baxter equation plays a crucial role in diverse problems in theoretical physics. These include exactly soluble models in statistical physics (Baxter, 1982) and quantum integrable model field theory (Faddeev, 1981; Bogoliubov *et al.*, 1985; Bullough *et al.*, 1980; Sklyanin, 1982; Kulish and Reshetkhin, 1983; de Vega *et al.*, 1984). Quantum groups provide a concrete example of noncommutative differential geometry (Connes, 1986). The idea of the quantum plane was first introduced by Manin (1987, 1988, 1989). The application of noncommutative differential geometry to quantum matrix groups was made by Woronowicz (1987, 1989). Wess and Zumino (1990; Zumino, 1991) considered one of the simplest examples of noncommutative differential calculus over Manin's quantum plane. They developed a differential calculus on the quantum hyperplane covariant with respect to the action of the quantum deformation of $GL(n)$, so-called $GL_q(n)$. Following this, much work has been done in this direction (Schmidke *et al.*, 1989; Giler *et al.*, 1991).

In spite of this work, it is uncertain whether this new mathematical object will bring forth new, "phenomena" in physics. Since symmetries play an important role in physics, it is worth extending them to the concept of deformed symmetries, which might be used in physics as well. Quantum groups applied to physics, are supposed to create a kind of "new" physics

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which goes back to its classical versions when the deformation parameters take particular values. To this end it is worthwhile to clarify the fundamental concepts and the computational techniques of quantum groups.

In this paper we obtain a q-analog of the contraction rule of the q-deformed Levi-Civita symbol, which is defined as

$$E_{12\ldots n} = 1 \tag{1}$$

and

$$E_{\ldots ij \ldots} = (-q) E_{\ldots ji \ldots} \qquad (i > j) \tag{2}$$

For example, the q-Levi-Civita symbol of rank three is easily computed according to the definition (1), (2):

$$\begin{aligned}
E_{123} &= 1 \\
E_{132} &= (-q)E_{123} = -q \\
E_{213} &= (-q)E_{123} = -q \\
E_{231} &= (-q)E_{213} = (-q)^2 E_{123} = (-q)^2 \\
E_{312} &= (-q)E_{132} = (-q)^2 E_{123} = (-q)^2 \\
E_{321} &= (-q)E_{231} = (-q)^2 E_{213} = (-q)^3 E_{123} = (-q)^3
\end{aligned}$$

When q goes to 1, the above equations reduce to 1 (or -1) for even (or odd) permutation of (1, 2, 3).

To begin with, we write down the q-deformed contraction rule of the q-deformed Levi-Civita symbol and prove it later:

$$E_{i_1 \ldots i_N k} E_{j_1 \ldots j_N k} = q^{2(\Sigma_{l=1}^{N} i_l - S(i_1, \ldots i_N) - N)} \sum_{\pi \in S_N} E^{j_1 \ldots j_N k}_{i_1 \ldots i_N k} \delta^{i_1}_{\pi(j_1)} \delta^{i_2}_{\pi(j_2)} \cdots \delta^{i_N}_{\pi(j_N)} \tag{3}$$

where S_N means the permutation group of degree N and

$$E^{j_1 \ldots j_N k}_{i_1 \ldots i_N k} = \frac{E_{i_1 \ldots i_N}}{E_{j_1 \ldots j_N}} \tag{4}$$

Here $S(i_1, \ldots, i_N)$ is defined as

$$S(i_1, \ldots, i_N) = \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} S(i_n, i_m) \tag{5}$$

where

$$\begin{aligned}
S(i, j) &= 1 \qquad \text{if} \quad i < j \\
S(i, j) &= 0 \qquad \text{if} \quad i \geq j
\end{aligned} \tag{6}$$

For example, $S(1,3,2,4)$ is computed as follows:

$$S(1,3,2,4) = S(1,3) + S(1,2) + S(1,4) + S(3,2) + S(3,4) + S(2,4) = 5$$

Now we will prove the property of the q-Levi-Civita symbol (3) by means of mathematical induction. Let us assume that equation (3) holds for the q-Levi-Civita symbol of rank N. First we can easily obtain an equivalent form of equation (3) as follows:

$$E_{i_1 \ldots i_N k} E_{j_1 \ldots j_N k} = Q^{2(\Sigma_{l=1}^{N} i_l - S(i_1, \ldots, i_N) - N)} \times \sum_{l=1}^{N} \delta_{j_l}^{i_1} \sum_{\pi_l \in S_{N-1}(\hat{j}_l)} E_{i_1 \ldots i_N k}^{j_1 \ldots j_N k} \delta_{\pi_l(j_1)}^{i_2} \delta_{\pi_l(j_2)}^{i_3} \cdots \delta_{\pi_l(j_N)}^{i_N} \quad (7)$$

where $S_{N-1}(\hat{j}_l)$ means the permutation group of degree $N-1$ where $\hat{j}_l$ is deleted.

Let us define $P(i,j)$ as

$$\begin{aligned} P(i,j) &= 1, \qquad i > j \\ P(i,j) &= -1, \qquad i < j \end{aligned} \quad (8)$$

Consider the case that $i_1 = j_l = I$, $I = 1, 2, \ldots, N$. From the definition of the q-Levi-Civita symbol we obtain

$$E_{i_1 \ldots i_N k} E_{j_1 \ldots j_N k} = (-q)^{2(I-1) + \Sigma_{k=1}^{l-1} P(j_k, j_l)} E_{i_2 \ldots i_N k} E_{j_1 \ldots \hat{j}_l \ldots j_N k} \quad (9)$$

Since we assumed that the q-contraction rule of the q-Levi-Civita symbol holds for the rank N case, we have

$$E_{i_1 \ldots i_N k} E_{j_1 \ldots j_N k} = (-q)^{2(I-1) + \Sigma_{k=1}^{l-1} P(j_k, j_l)} q^{2(\Sigma_{l=2}^{N} i_l + I - N - S(i_2, \ldots, i_N) - (N-1))} \times \sum_{\pi_l \in S_{N-1}(\hat{j}_l)} E_{i_2 \ldots i_N k}^{j_1 \ldots \hat{j}_l \ldots j_N k} \delta_{\pi_l(j_1)}^{i_2} \delta_{\pi_l(j_2)}^{i_3} \cdots \delta_{\pi_l(j_N)}^{i_N} \quad (10)$$

Using the relation

$$E_{i_2 \ldots i_N k}^{j_1 \ldots \hat{j}_l \ldots j_N k} = (-q)^{-\Sigma_{k=1}^{l-1} P(j_k, j_l)} E_{i_1 \ldots i_N k}^{j_1 \ldots j_N k} \qquad \text{for} \quad i_1 = j_l = I \quad (11)$$

we have

$$E_{i_1 \ldots i_N k} E_{j_1 \ldots j_N k} = q^{2(\Sigma_{l=1}^{N} i_l - (S(i_2, \ldots, i_N) + N - I) - N)} \times \sum_{\pi_l \in S_{N-1}(\hat{j}_l)} E_{i_1 \ldots i_N k}^{j_1 \ldots j_N k} \delta_{\pi_l(j_1)}^{i_2} \delta_{\pi_l(j_2)}^{i_3} \cdots \delta_{\pi_l(j_N)}^{i_N} \quad (12)$$

From the definition of $S(i_1, \ldots, i_N)$, we have for $i_1 = j_l = I$

$$S(I, i_2, \ldots, i_N) = \sum_{m=2}^{N} S(I, i_m) + S(i_2, \ldots, i_N)$$

$$= N - I + S(i_2, \ldots, i_N) \tag{13}$$

Inserting equation (13) into the right-hand side of equation (12), we complete the proof of equation (3) by the induction principle.

In particular, for the $N = 2$ case, equation (3) takes the form

$$E_{ijk}E_{lmk} = \theta(j-i)q^{2(i+j-3)}[\delta^i_l\delta^j_m - q\delta^i_m\delta^j_l]$$

$$+ \theta(i-j)q^{2(i+j-2)}[\delta^i_l\delta^j_m - q^{-1}\delta^i_m\delta^j_l] \tag{14}$$

There exists another contraction rule for the q-Levi-Civita symbol:

$$E_{ijk}E_{klm} = \theta(j-i)q^2[\delta^i_l\delta^j_m - q\delta^i_m\delta^j_l]$$

$$+ \theta(i-j)q^4[\delta^i_l\delta^j_m - q^{-1}\delta^i_m\delta^j_l] \tag{15}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be q-vectors in three-dimensional quantum space R^3_q. Then we have

$$\mathbf{A} = (A_1, A_2, A_3)$$

$$\mathbf{B} = (B_1, B_2, B_3)$$

We define the ith component of the q-deformed outer product of two q-vectors $\mathbf{A}$ and $\mathbf{B}$ as

$$(\mathbf{A} \times \mathbf{B})_i = E_{ijk}A_jB_k \tag{16}$$

Then we obtain the q-analog of the $BAC - CAB$ rule for the q-deformed outer product:

$$(\mathbf{A} \times (\mathbf{B} \times \mathbf{C}))_i = q^2B_i\langle\mathbf{A}, \mathbf{C}\rangle_{q|i} - q^3C_i\langle\mathbf{A}, \mathbf{B}\rangle \tag{17}$$

where

$$\langle\mathbf{A}, \mathbf{C}\rangle_{q|i} = \sum_{i<j} A_jC_j + qA_iC_i + q^2\sum_{j<i} A_jC_j \tag{18}$$

$$\langle\mathbf{A}, \mathbf{B}\rangle = \sum_j A_jB_j \tag{19}$$

Proof. From the definition of the q-outer product we have

$$\begin{aligned}
(\mathbf{A}\times(\mathbf{B}\times\mathbf{C}))_i &= E_{ijk}A_j(\mathbf{B}\times\mathbf{C})_k \\
&= E_{ijk}E_{klm}A_jB_lC_m \\
&= \theta(j-i)q^2(\delta^i_l\delta^j_m - q\delta^i_m\delta^j_l)A_jB_lC_m \\
&\quad + \theta(i-j)q^4(\delta^i_l\delta^j_m - q^{-1}\delta^i_m\delta^j_l)A_jB_lC_m \\
&= \sum_{i<j} q^2(A_jB_iC_j - qA_jB_jC_i) \\
&\quad + \sum_{j<i} q^4(A_jB_iC_j - q^{-1}A_jB_jC_i) \\
&= B_i\left(\sum_{i<j} q^2A_jC_j + \sum_{j<i} q^4A_jC_j\right) \\
&\quad - q^3C_i\left(\sum_{i<j} A_jB_j + \sum_{j<i} A_jB_j\right) \\
&= q^2B_i\langle\mathbf{A},\mathbf{B}\rangle_{q|i} - q^3C_i\langle\mathbf{A},\mathbf{B}\rangle
\end{aligned}$$

In this paper we have obtained a q-analog of the contraction rule of the q-Levi-Civita symbol and proved it. We think that this formula could be applied to the deformation of geometric quantization, q-deformed Maxwell or Yang–Mills theory, and q-deformed Einstein theory. We hope that this formula will be widely used in developing q-deformation physics.

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